Does it take longer for a cricket ball to reach its maximum height when hit by a bat or to return to Earth from its maximum height?

Math AA HL Internal Assessment	**Examination Session: 2023** **Number of pages used: 17**

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Introduction

Rationale

I came across a post while surfing on the internet (appendix 1), this post had a debate pole if a ball would take longer to go up or come down and 56% of the respondents responded it would take the same time while the other 27% said down and the rest of them said up, this got me questioning what actually the answer is. The comment section had very mixed opinions which got me more curious to model this project. I tried doing this practically although doing this practically was very difficult which got me questioning if I could mathematically model this. Since I love watching and playing cricket, and Cricket is a sport where most of the time the ball is hit and it goes up or down, I used a cricket ball's mass and the speed of a ball hit by a bat's speed. According to my speculation, I think the ball will take the same time to go up and come back down. This begs the question, "Does it take longer for a cricket ball to reach its maximum height when hit by a bat or to return to Earth from its maximum height?" In this investigation, I'll look at the time it takes the ball to rise and the time it takes the ball to fall.

Aim

The aim of this exploration is to determine whether a cricket ball will take longer to reach maximum height when hit by a bat or return back to Earth from its maximum height.

Methodology

The mathematics Involves modelling the motion of the model through a differential equation by utilizing Newton's Second Law of Motion to find the overall force acting on the ball (differential equations provide a mathematical framework for describing how the rate of change of a system depends on its current state which allows us to make predictions about the future behavior of the system based on its initial conditions), as well as modelling the air resistance as a linear function of velocity of the ball. The differential equation I am going to derive for the motion of a ball being thrown directly up in the air, and solving it in terms of the velocity of the ball as a function of time. The solution will involve solving a separable differential equation (the reason for using separable equation is so that we can obtain an equality between two integrals, This form will allow me to isolate the variables and integrate both sides, which will lead to finding a general solution for the equation) (Separable Equations Introduction, 2022) with the initial value being known. Then I will derive the formula for the height of the ball being thrown directly in the air as a function of time, Using the derivation of the equation for the velocity of the ball, we can use this formula to determine the height of the ball. Recall that the velocity is simply the derivative of the displacement function, in this case the position function is the height function. Thus, we can take the integral of the velocity to obtain the position function

Then we solve for the time it takes for the ball to reach its maximum height. This is solved by the fact that at the maximum height, the velocity of the ball is zero because it is not moving just before it starts falling back down to the earth. Since we know the formula for the velocity of the ball from before, we can solve for the time to reach its maximum height. Also in this question, I solve for this specific time for the case, I can take the ball's weight as 0.2 kg which is the average mass of a cricket ball, the initial velocity is 80 m/s as it is the average speed a ball is hit at (Cricket Ball Speed off Bat, 2022), and the air resistance is a tenth of the speed of the ball.

Using the mass and average speed a cricket ball is hit at we can determine the time it takes for it to reach the ground after its maximum height. I shall graph the equations with values and get the time it takes for the ball to go up and come down.

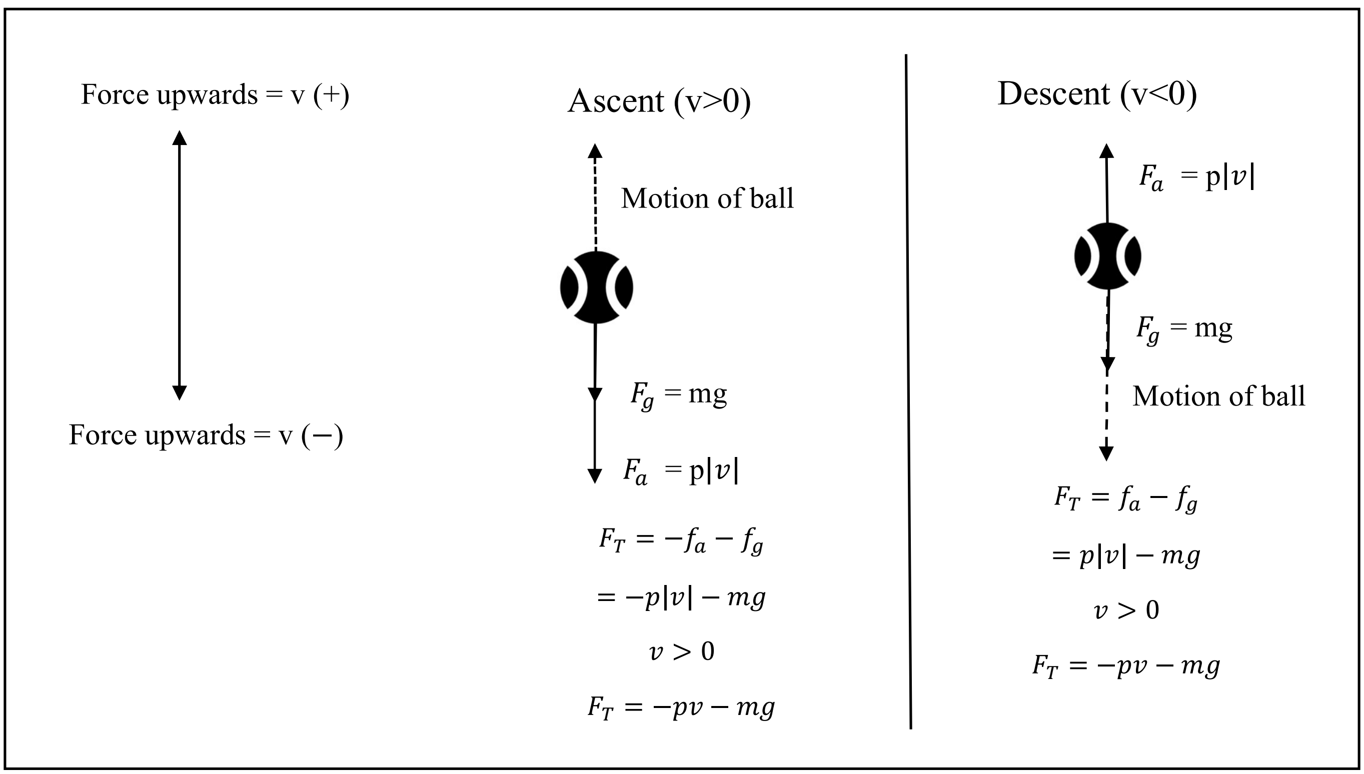
For the calculations I will use a GDC, and for graphing I will use Desmos as it is accurate and user-friendly. I will be using 3 significant figures throughout my investigation.

Deducing the equation of velocity

Variables used in the IA		
Quantity	Symbol	Units
Mass	m	Kg
Initial velocity	V_0	ms^{-1}
Positive air resistance constant	p	N
Velocity	v	ms^{-1}
Time	t	s
Gravitational Force	F_g	N
Air resistance	F_a	N
Total Force	F_T	N

Table 1: Showing the variables used in the IA

We Assume that the vertically upward projected mass of the ball has a positive initial velocity from the Earth's surface. We also assume that the forces acting on the ball are gravity and the opposing air resistance, both of which have magnitudes of $p|v(t)|$.



(Diagram 1: Shows the force acting on the ball in ascent and descent)

The total force applied on the ball during the ascent and descent is $-pv - mg$.

As the object ascents, the resistance acts downward and v(t) is positive.

- As the object descends, the resistance acts upward and v(t) is negative.

Thus, the equation of motion according to Newton's Second Law is:

$$F_T = ma = mv' = -pv - mg$$

$$m\frac{dv}{dt} = -pv - mg \text{ and } v(0) = v_0$$

$$dv = -(pv + mg)\frac{dt}{m}$$

$$\frac{dv}{pv + mg} = -\frac{dt}{m}$$

Then integrate both sides

$$\int \frac{dv}{pv + mg} = \int -\frac{dt}{m} = \frac{-t}{m} + c_2$$

$$let\colon u = pv + mg$$

$$du = pdv \to dv = \frac{du}{p}$$

$$\int \frac{dv}{pv + mg} = \int \frac{1}{u}du = \frac{1}{p}\ln|u| + c_1$$

$$\frac{1}{p}\ln|pv + mg| + c_1 = \frac{-t}{m} + c_2$$

From diagram 1, Assume $mg \geq p|v|$ because otherwise the ball will slow down and eventually will start floating upwards. $mg = p|v|$ →terminal velocity.

$$mg = p|v| \rightarrow mg - p|v| \geq 0$$

$$|mg + pv| = mg + pv \geq 0$$

Hence we can remove the absolute value sign because when the ball is going upwards $v > 0$ which means $pv + mg > 0$, hence there is no need of an absolute value sign as the value is going to be greater than 0 which means it won't be negative.

$$\frac{1}{p}\ln(pv + mg) = \frac{-t}{m} + c \quad \text{where: } c = c_2 - c_1 = constant$$

From our initial condition $v(0) = v_0$

$$\frac{1}{p}\ln(pv_0 + mg) = -\,0 + c = c$$

After combining both equations we get:

$$\frac{1}{p}\ln(pv + mg) = \frac{-t}{m} + \frac{1}{p}\ln(pv_0 + mg)$$

$$\ln(pv + mg) = \frac{-pt}{m} + \ln(pv_0 + mg)$$

$$e^{\ln(pv+mg)} = e^{\left(\frac{-pt}{m} + \ln(pv_0+mg)\right)}$$

$$pv + mg = e^{\frac{-pt}{m}} \times (\,pv_0 + mg)$$

Then we divide by p

$$pv = e^{\frac{-pt}{m}} \times (pv_0 + mg) - mg$$

Hence we get the velocity equation:

$$v(t) = \left(v_0 + \frac{mg}{p}\right)e^{\frac{-pt}{m}} - \frac{mg}{p}$$

Equation for height of the ball, until it hits the ground:

Velocity is the derivative of the position function(height) so the velocity is the derivative of the height.

$$v(t) = \frac{dy(t)}{dt}$$

$$\int v(t)dt = \int dy(t) = y(t)$$

$$y(t) = \int v(t)dt$$

When $(t = 0)$ we assume the ball is on the ground, hence height is also 0.

$$y(t = 0) = 0$$

From the above velocity equation

$$= \int \left[\left(v_0 + \frac{mg}{p}\right) e^{\frac{-pt}{m}} - \frac{mg}{p}\right] dt$$

$$y(t) = \left(v_0 + \frac{mg}{p}\right)\left(\frac{m}{-p}\right) e^{\frac{-p}{m}} - \frac{mg}{p} t + c$$

From initial condition when $y(0) = 0$

$$0 = -\left(v_0 + \frac{mg}{p}\right)\frac{m}{p}\ (1) - 0 + c$$

$$c = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}$$

Then we substitute the c obtained into the $y(t)$ equation.

$$y(t) = -\left(v_0 + \frac{mg}{p}\right)\left(\frac{m}{p}\right) e^{\frac{-pt}{m}} - \frac{mg}{p} t + \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}$$

Hence we get the height equation:

$$y(t) = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}\left(1 - e^{\frac{-p}{m}}\right) - \frac{mgt}{p}$$

Equation for the time it takes the ball to reach its maximum height

Assume that t_1= time it takes ball to reach maximum height

When the ball is in the air the height is maximum and the velocity is 0

Diagram 2 showing when height is maximum the velocity is 0

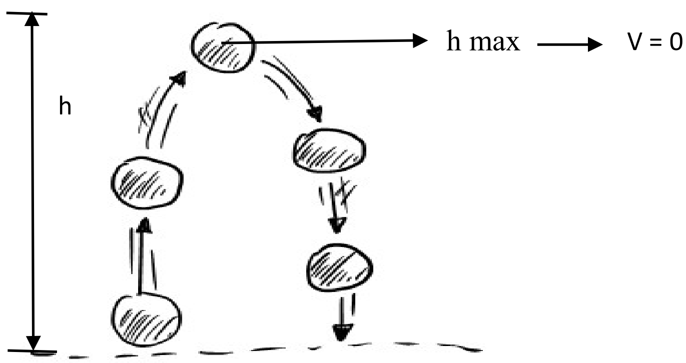


Diagram 2

Using the velocity equation from above, and using $v(t_1) = 0$

$$0 = \left(v_0 + \frac{mg}{p}\right)e^{\frac{-pt}{m}} - \frac{mg}{p}$$

$$\frac{mg}{p} = \left(v_0 + \frac{mg}{p}\right)e^{\frac{-pt}{m}}$$

$$\frac{mg}{p} \times \left(\frac{1}{v_0 + \frac{mg}{p}}\right) = e^{\frac{-p}{m}}$$

Apply ln both sides

$$\ln\left(\frac{mg}{p} \times \frac{1}{v_0 + \frac{mg}{p}}\right) = \ln\left(e^{\frac{-p}{m}}\right)$$

$$\ln\left(\frac{mg}{p}\times\frac{1}{v_0+\frac{mg}{p}}\right)=\frac{-pt}{m}$$

Multiply $negative(-)$ both side

$$-\ln\left(\frac{mg}{p}\times\frac{1}{v_0+\frac{mg}{p}}\right)=-\frac{-pt}{m}$$

$$\ln\left(\frac{mg}{p}\times\frac{1}{v_0+\frac{mg}{p}}\right)^{-1}=\frac{pt}{m}$$

$$\ln\left(\frac{p}{mg}\times\left(v_0+\left(\frac{mg}{p}\right)\right)\right)=\frac{pt_1}{m}$$

Hence we get a time equation

$$t_1=\frac{m}{p}\times\ln\left(\frac{mg+pv_0}{mg}\right)$$

Finding time t_1 it takes for the ball to reach maximum height

Approximate values used for the symbols.		
Quantity	**Symbol**	**Amount**
Mass	m	0.200g
Initial Velocity	v_0	$80.0ms^{-1}$
Gravitational force	g	$9.81ms^{-2}$
Air resistance ($\frac{1}{10}$ of the speed)	p	$F_a=\frac{1}{10.0}\lvert v(t)\rvert=p\lvert v(t)\rvert\rightarrow p=\frac{1}{10.0}$

(Table 2: Shows approximate values used for the symbols.)

Using the data in table 2:

$$t_1=\frac{0.200}{\left(\frac{1}{10.0}\right)}\ln\left(\frac{0.200\times 9.81+\frac{1}{10.0}\times 80.0}{0.200\times 9.81}\right)$$

$$t_1 = 2 \times \ln\left(\frac{9.962}{1.962}\right)$$

$$t_1 = 3.24963\ seconds$$

$$t_1 \approx 3.25\ seconds$$

$$(Answer\ obtained\ Using\ G.D.C)$$

Finding time (t_2) it takes for the ball to fall back to earth

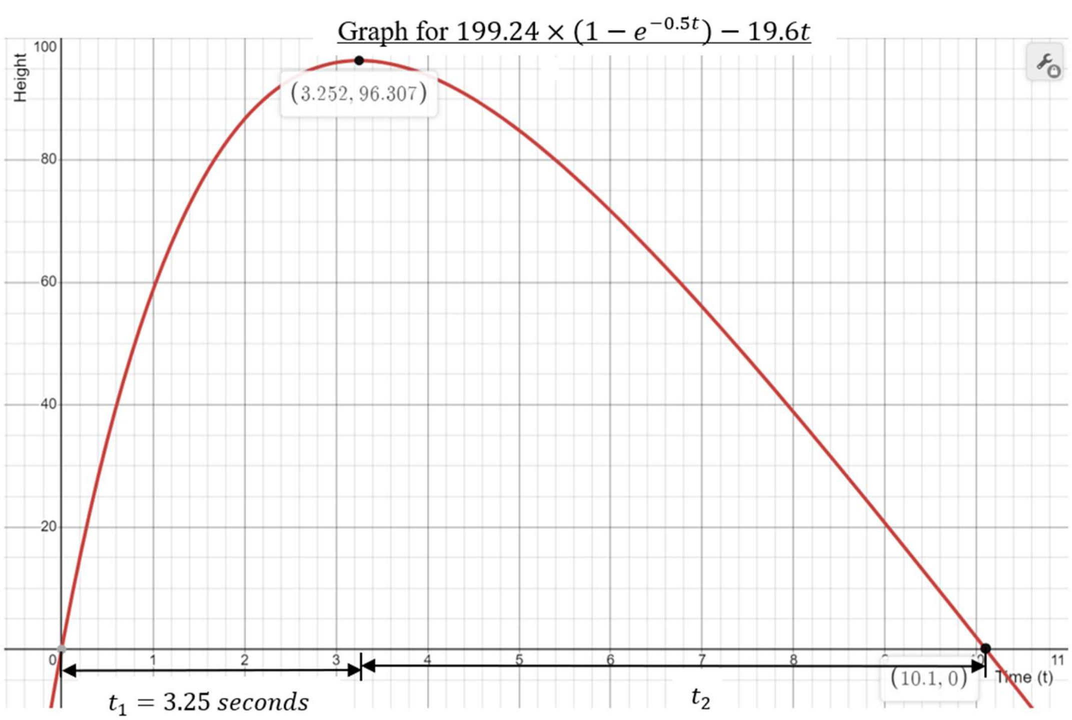
We can estimate t_2 by using a graph of the height function y(t)

Using the data in table 1:

$$y(t) = \left(\left(80 + \frac{0.2 \times 9.81}{\frac{1}{10}}\right)\frac{0.2}{\frac{1}{10}}\right) \times \left(1 - e^{\frac{-\frac{1}{10}t}{0.2}}\right) - \left(\frac{0.2 \times 9.81 \times t}{\frac{1}{10}}\right)$$

$$y(t) = 199.2 \times (1 - e^{-0.5t}) - 19.6t$$

Then graph the above equation



From the graph above we can see that the total time is 10.1 hence we can find t_2

$$t_2 = 10.1 - 3.25 = 6.85\ seconds$$

Hence t_2 is about $6.85 - 3.25 = 3.60$ seconds longer

Finding if it goes up faster or returns faster without any real value

I consider the general case of physical properties and evaluate if a ball reaches its maximum height faster or slower than it does after reaching its maximum height. We can solve for time indirectly because we can't solve it explicitly from the height equation. This is done by determining whether the height function is positive or negative (or zero) when the time is equal to two times the time required to achieve the maximum height.

Diagram 3: Shows a graph comparing time of ball going up and coming down

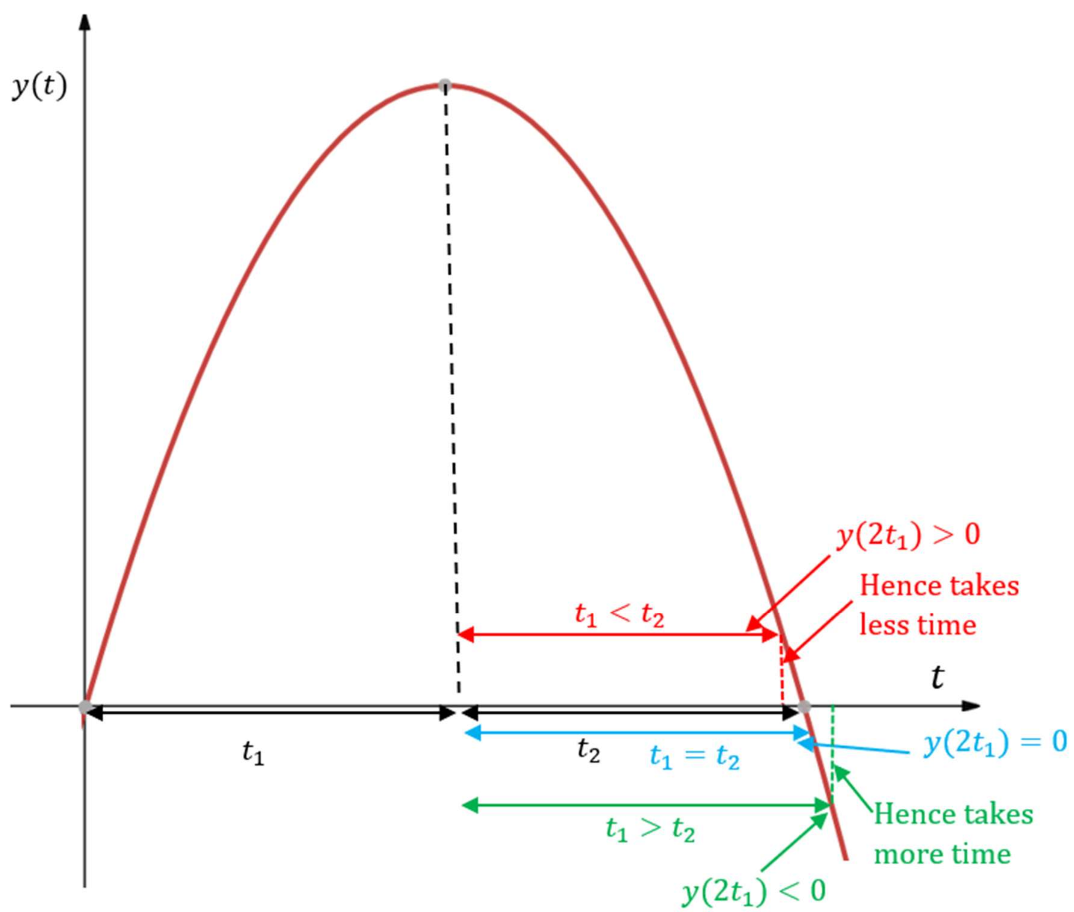


Diagram 3

If t_1 is less than t_2, $y(2t_1)$ is positive as it is greater than 0 which means the ball is still in the air.

If t_1 is equal to t_2, $y(2t_1)$ is equal to 0.

If t_1 is more than t_2, $y(2t_1)$ is negative as it is less than 0, which means the ball has a negative height if it takes longer.

It is impossible to find t_2 because equating $y(t) = 0$ is not possible. This is because we can't solve for the time explicitly from the height equation.

However, we may use an alternative way to evaluate whether ascending or descending is faster: we assess whether the time $t_1$and after it gets doubled $y(2t_1)$ is positive or negative.

Recalling the time equation:

$$t_1 = \frac{m}{p} \times \ln\left(\frac{mg + pv_0}{mg}\right)$$

Recalling the height equation:

$$y(t) = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}\left(1 - e^{\frac{-pt}{m}}\right) - \frac{mgt}{p}$$

Replace t with the equation of time

$$y(2t_1) = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}\left(1 - e^{2\times\frac{p}{m}\left[\frac{m}{p}\times\ln\left(\frac{mg+pv_0}{mg}\right)\right]}\right) - \frac{2mg}{p}\left[\frac{m}{p} \times \ln\left(\frac{mg + pv_0}{mg}\right)\right]$$

Let $x = e^{\frac{p}{m}\left[\frac{m}{p}\times\ln\left(\frac{mg+pv_0}{mg}\right)\right]}$

$$= e^{\frac{\cancel{p}}{\cancel{m}}\left[\frac{\cancel{m}}{\cancel{p}}\times\ln\left(\frac{mg+pv_0}{mg}\right)\right]}$$

$$= \frac{mg + pv_0}{mg}$$

Take $x = \frac{mg+pv_0}{mg}$

$$e^{-2\times\frac{p}{m}\left[\frac{m}{p}\times\ln\left(\frac{mg+pv_0}{mg}\right)\right]} = \left(e^{\frac{p}{m}\left[\frac{m}{p}\times\ln\left(\frac{mg+pv_0}{mg}\right)\right]}\right)^{-2} = x^{-2}$$

Solving further

$$\frac{pv_0}{p} + \frac{mg}{p} = \left(\frac{pv_0 + mg}{p}\right)\frac{mg}{mg}$$

$$= \left(\frac{mg + pv_0}{mg}\right)\frac{mg}{p} = x \cdot \frac{mg}{p}$$

Hence

$$y(2t_1) = (x)\left(\frac{mg}{p}\right)\left(\frac{m}{p}\right)(1 - x^{-2}) - 2\frac{m^2}{p^2}g \cdot \ln x$$

$$= \frac{m^2}{p^2}gx \cdot \left(1 - \frac{1}{x^2}\right) - 2\frac{m^2}{p^2}g \cdot \ln x$$

$$y(2t_1) = \frac{m^2}{p^2}g\left(x - \frac{1}{x} - 2\ln x\right)$$

Showing $x > 1$

$$x = e^{\frac{p}{m}\left[\frac{m}{p}\times\ln\left(\frac{mg+pv_0}{mg}\right)\right]}$$

p is a positive constant hence it is greater than 0, t_1 is greater than 0 because time can't be negative, m is greater than 0 because we can't have a negative mass.

$$p > 0 \quad t_1 > 0 \quad m > 0$$

$$= \frac{p}{m}\left[\frac{m}{p} \times \ln\left(\frac{mg + pv_0}{mg}\right)\right] > 0$$

$$x > e^0 = 1$$

$$x > 1$$

Showing $f(x)$ is increasing for $x > 1$

$$f(x) = x - \frac{1}{x} - 2\ln x$$

$$f'(x) = 1 - (-x^{-2}) - \frac{2}{x}$$

$$f'(x) = \frac{x^2 - 2x + 1}{x^2}$$

It is greater than 0 because any value we replace x with the answer will still be greater than 0

$$f'(x) = \frac{(x-1)^2}{x^2} > 0$$

Hence $f(x)$ is increasing for $x > 1$

$$y(2t_1) = \frac{m^2}{p^2} g \cdot f(x)$$

If $x = 1$

$$f(1) = 1 - \frac{1}{1} - 2\ln 1$$

$$f(1) = 0$$

Since $y(2t_1) > 0$, t_1 is less than t_2.Thus Ascent is faster than descent.

Conclusion and evaluation

Hence I can conclude that a ball goes up slightly faster than it comes down. This variation was caused by the inclusion of air resistance in the model. If this motion had been performed in a vacuum, the trajectory would have been parabolic, and the ball would have taken the same amount of time to go up and down. This has especially helped me as I used to think the amount of time it takes to go up is the same as it takes to come down.

This result makes sense because air resistance always acts against the motion of the ball, whereas gravity's force is constant and always acts downward. So, on the way up, there are two downward forces acting on the ball, limiting its overall vertical rise. On the way down, there is one upward force that increases as the ball speeds up (air resistance) and one downward force that remains constant (gravity).

Strength

The strength of my investigation is I used a mathematical approach to describe the motion of a cricket ball while accounting for many aspects including air resistance and the ball's beginning velocity. I derived a general solution for the motion of the ball and used it to make predictions about its behavior. I also made sure that my investigation was applicable to a specific scenario of a cricket ball being struck by a bat by mentioning the mass and average speed of a cricket ball. I imagined the ball's motion and gained a better understanding of how long it takes for it to rise to its highest point and fall back to the ground by graphing the equations with values.

Overall, my investigation is made more thorough and accurate by the application of mathematical modeling and statistics to the specific scenario of a cricket ball being hit by a bat.

Limitations

The limitation associated with my investigation are the value of air resistance, it was difficult to find the air resistance acting on the ball as it would need the air density, drag coefficient and velocity at different interval which would make this modelling very difficult and complicated. Additionally the only forces taken against the ball were weight and air resistance, there are way more forces acting on the ball for example lift which can possibly alter the results, though the result of the ball taking longer to fall back from maximum height to the earth still stay the same. The value obtained from the graph is only theoretically correct because we assume the air resistance is $\frac{1}{10}th$ of velocity, this means that the air resistance acting on the ball is not particularly accurate. Another limitation of this question is that it assumes a simplified model of the flight of a cricket ball. In reality, there are many factors that can affect the trajectory of a ball, such as wind conditions and spin. Moreover, this question does not take into account the specific circumstances of each cricket match, such as the skill of the players, the condition of the ball and pitch, and the tactics of the teams. These factors can greatly influence the trajectory of a ball and make it difficult to provide a definitive answer to the question. Finally, this question may not be relevant to all students, as it is specific to the game of cricket. Students who are not familiar with the rules and dynamics of the game may struggle to understand the context of the question and its implications.

Extension

Personally, I think it would be a fascinating continuation of this research to account for the majority of the pressures operating on the ball in addition to identifying and accounting for any actual or potential air resistance.

The type of ball being used has an impact in addition to the previously mentioned elements on how long it takes a cricket ball to reach its highest point and return to Earth. A cricket ball can be made of leather or synthetic materials, and each has unique characteristics that influence its flight. A leather ball swings higher in the air and is heavier than a synthetic ball. Because of its weight and the swing it produces, a leather ball may therefore take longer to reach its maximum height when struck by a bat. Its weight and increased momentum could cause it to fall back to the ground more quickly than a synthetic ball, though. A synthetic ball, on the other hand, has a higher trajectory because it is lighter and tends to bounce more on the field. As a result, when struck by a bat, a synthetic ball could reach its maximum height more quickly than a leather ball. On the other hand, because of its smaller weight and the reduced gravitational pull on it, it might take longer to fall back to the ground.

The speed and angle of the hit, the type of ball used, air resistance, and gravitational force are all factors that affect how long it takes a cricket ball to reach its maximum height and return to Earth.

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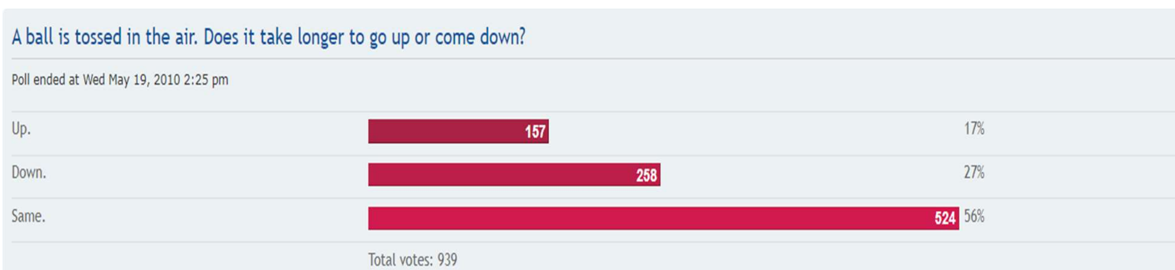
Appendix



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